

# Group Velocity in Strongly Scattering Media

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Investigation of the ballistic propagation of acoustic waves through a resonantly scattering, inhomogeneous medium indicates that although the ballistic signal remains coherent with the incident pulse, it is nevertheless strongly affected by scattering resonances. These resonances cause considerable frequency dispersion and substantially reduce the phase and group velocities. The experimental data are quantitatively described by a theoretical model that correctly accounts for the coupling between the resonant scatterers, leading to an effective renormalization of the scattering within the medium. This approach resolves a long-standing problem in the definition of the group velocity in strongly scattering materials.

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Virtually all forms of energy are propagated by waves: heat and sound in the form of acoustic waves, and radio and light in the form of electromagnetic waves. In any medium, wave propagation depends on the relation of the angular frequency of a wave,  $\omega$ , to its wave vector,  $k$ , as given by the dispersion relation  $\omega(k)$ . Whenever the dispersion relation is nonlinear, waves of different frequencies travel at different speeds and typically require two distinct velocities to describe wave propagation: The first is the phase velocity,  $v_p = \omega/k$ , the speed at which a plane of constant phase propagates; the second is the group velocity,  $v_g = d\omega(k)/dk$ , normally the speed at which a

pulse or signal propagates. However, the meaning of the group velocity is strictly well defined only when the dispersion of the medium is not too large (1). Nevertheless, even in a highly dispersive medium, a signal can still propagate, and the determination of its speed of propagation is a classic problem, having been recognized in the work of Sommerfeld (2) and Brillouin (3).

One important manifestation of this problem is the description of wave propagation through strongly scattering, inhomogeneous materials (4). In the intermediate-frequency regime, where the length scale of the inhomogeneities is comparable to the wavelength of the wave, the excitation of

internal modes within the scatterers leads to strong resonant scattering. This scattering results in a large attenuation of the propagating wave, even in the absence of absorption, and a correspondingly large dispersion. The dispersion may lead to unphysical values of the group velocity (5), supporting the ideas of Sommerfeld and Brillouin that the group velocity completely loses its meaning in these circumstances. However, even in the most strongly scattering medium, a coherent, ballistic portion of a transmitted pulse can always be measured if one uses a sufficiently thin sample. This ballistic pulse consists of both the unscattered and forward scattered portions of the wave pulse.

What is the velocity at which this pulse propagates? This question is intriguing because, on the one hand, it does not seem possible for the signal to propagate through the medium without being affected by the scattering, yet on the other hand, if it is scattered, temporal coherence between the transmitted and incident pulses may be destroyed. If the pulse is not scattered, its velocity should be intermediate between the velocities of the individual components of the medium. By contrast, if the scatterers do affect the ballistic propagation, the resultant dispersion may become very large, making the group velocity inadequate to describe ballistic pulse propagation. This question has remained an unresolved dilemma, despite the enormous recent progress in understanding the multiple scattering, diffusion, and possible localization of classical waves (4). In this report, we resolve this basic dilemma.

We focused on ultrasonic waves and measured both the phase and group velocities of the ballistic signals that were transmitted through thin slabs of a strongly scattering material. Strong resonant scattering significantly modified both velocities, making them slower than any of the velocities of the individual components that comprise the medium. We quantitatively calculated both velocities using a generalized effective-medium theory of excitations in the strong scattering limit (6, 7). The same theory also

accurately predicts the measured frequency dependence of the scattering mean free path, establishing a direct link between the dispersion and the excitation of internal resonances within the individual scatterers. The agreement between the theory and experiment provides insight into the origin of the pronounced effects of the strong scattering on the ballistic propagation of sound. It indicates that the coupling between the strong resonant scatterers results in an effective renormalization of the medium through which the ballistic pulse propagates, thereby showing explicitly how coherent wave propagation is influenced by the scattering. The group and phase velocities are significantly reduced but remain well defined and physically meaningful.

The samples were disc-shaped slabs of relatively monodisperse glass beads, randomly packed in water at a glass-bead volume fraction of about 63% and contained between two 1.5-mm-thick sheets of polystyrene. The thickness of the samples ranged from about 2 to 5 mm. Each sample was immersed in water, with a 25-mm-diameter flat transducer generating the sound on one side of the slab and a second, identical receiving transducer on the other side. The frequency of the sound was varied between 1 and 5 MHz, and samples with bead radii of  $a = 0.25$  and  $0.5$  mm were used to increase the effective frequency range. The velocities of ballistic propagation were measured with a square input pulse consisting of about 10 oscillations. Because the detector averages the instantaneous pressure of the acoustic wave over the front face of the transducer, the scattered sound was effectively canceled as a result of its random phase fluctuations in the plane of the detector, reflecting the presence of speckles. This destructive interference left only the unscattered ballistic wave, which was spatially and temporally coherent across the entire surface of the detector. The phase cancellation of the scattered sound was further improved by translating the sample and averaging the transmitted field from different ensembles of the scatterers, thereby ensuring that the background contribution from scattered sound was eliminated from the ballistic pulse.

To determine the phase and group velocities, we used ultrasonic techniques to measure directly the phase and amplitude of the detected signal. The phase velocity was determined from the propagation time of the individual oscillations near the center of the pulse, where the carrier frequency is best defined. The correct correspondence between the oscillations of the incident and transmitted pulses was identified by ensuring consistency of the phase velocity between samples of several different thicknesses. The group velocity was determined

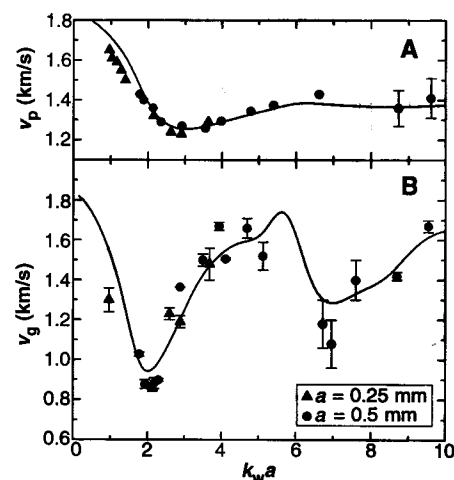
by matching the centers of the transmitted and incident wave forms and measuring the propagation time. We also digitally filtered both the incident and transmitted pulses to limit the signal to a Gaussian pulse with a relatively narrow bandwidth and determined the propagation time of the peak of the pulse. This improved the accuracy of the measurement; however, both techniques yielded essentially the same results.

Significant dispersion was observed (Fig. 1), with the phase velocity dropping by about 50% to a minimum value of 1.25 km/s at  $k_w a \approx 3$  ( $k_w$  is the acoustic wave vector in water), and with the group velocity varying by about a factor of 2, from a minimum of 0.85 km/s near  $k_w a \approx 2$  to a maximum of 1.7 km/s near  $k_w a \approx 4.5$ . Remarkably, for most of the frequencies at which these measurements were made, both velocities are substantially less than the speed of sound in either of the constituent materials, that is, either the longitudinal velocity in water ( $v_w = 1.5$  km/s) or the longitudinal ( $v_L = 5.6$  km/s) or transverse velocity ( $v_T = 3.4$  km/s) in the glass. Moreover, they are also less than the velocity of interfacial Stoneley waves (8) propagating on a flat interface between glass and water ( $v_S \approx 1.5$  km/s).

To calculate these velocities theoretically, we adopted a technique that has been used to determine the dispersion curves for sound propagation through suspensions of solid plastic spheres (6, 7, 9). We begin with the scalar wave equation

$$\left(\nabla^2 + \frac{\omega^2}{v_0^2}\right)\phi + \omega^2[v^{-2}(\mathbf{r}) - v_0^{-2}]\phi = 0 \quad (1)$$

where  $\phi$  denotes the wave amplitude, and  $v(\mathbf{r})$  is the local phase velocity, defined by the material parameters at position  $\mathbf{r}$ . In the



**Fig. 1.** (A) Phase and (B) group velocities of acoustic waves in a strongly scattering medium consisting of glass beads in water. The data (symbols) are compared with the predictions of the effective medium theory (lines).

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frequency and wave vector representation, the Green's function for Eq. 1, in which the scattering is defined relative to a uniform reference medium with velocity  $v_0$ , is given by

$$G(\omega, k) = \frac{1}{\omega^2 - k^2 - \Sigma_{v_0}} \quad (2)$$

where the self-energy  $\Sigma_{v_0}$  represents all of the effects of multiple scattering caused by the inhomogeneities, given by the term  $\omega^2[v^{-2}(\mathbf{r}) - v_0^{-2}]$  in Eq. 1. A general approach for obtaining the dispersion relation of excitations in a medium with strong scattering is based on the spectral function given by the negative imaginary part of the Green's function  $-\text{Im } G(\omega, k)$ . The zeroes of the denominator of the Green's function give the excitations of the system, and the calculation of  $-\text{Im } G(\omega, k)$  is one way of delineating this quantitatively. In the low-frequency regime, where the wavelength is much larger than the scale  $R$  of the inhomogeneities, there are true zeroes of the denominator in Eq. 2 to leading order of the small parameter  $R/\lambda$ , where  $\lambda$  is the wavelength. These zeroes are in fact the solution of the coherent-potential approximation (CPA) for the calculation of effective medium wave properties (10). However, in the intermediate frequency regime, there are no longer true zeroes; instead, there are only peaks in the function  $-\text{Im } G(\omega, k)$ . Thus, the CPA approach, at least in its traditional form, can no longer be applied.

To go beyond this limitation, we calculated the spectral function by exploiting the fact that velocity  $v_0$  enters only as a dummy variable in Eqs. 1 and 2, enabling it to be chosen as a free parameter in the calculations (6, 7, 9): As long as the value of  $v_0$  used in the calculations of  $\Sigma_{v_0}$  is the same as the value appearing in  $\omega^2/v_0^2$ , the Green's function is independent of the choice of  $v_0$ . By letting  $v_0 = \omega/k$  at every point in the  $\omega k$  plane and calculating the corresponding values of  $\Sigma_{v_0}$ , we can calcu-

late the spectral function using

$$-\text{Im } G(\omega, k) = \text{Im} \frac{1}{\Sigma_{\omega/k}} \quad (3)$$

This provides a full two-dimensional picture of the frequency and wave vector dependence of the spectral function. Because  $\Sigma_{v_0}$  represents the effect of the scattering, Eq. 3 tells us that peaks in the spectral function correspond to minima in the scattering. To accurately calculate the self-energy, one must take into account the strong geometric correlation that exists in our random medium, because it is comprised of spheres dispersed in a fluid. To do this, we modeled a typical scatterer within the medium as a sphere coated by a layer of fluid. The thickness of this coating was determined by the volume fraction of the spheres. This coated sphere is embedded in a homogeneous medium whose wave speed is  $v_0 = \omega/k$ . The self-energy is then obtained by solving the boundary value problem for a plane wave scattering from the embedded coated sphere (7).

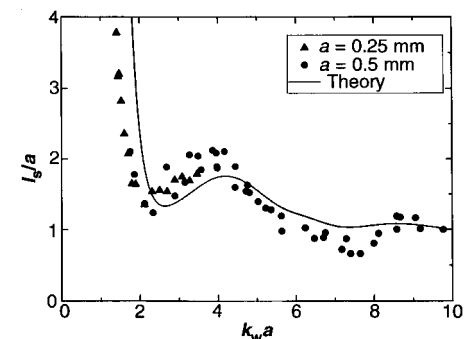
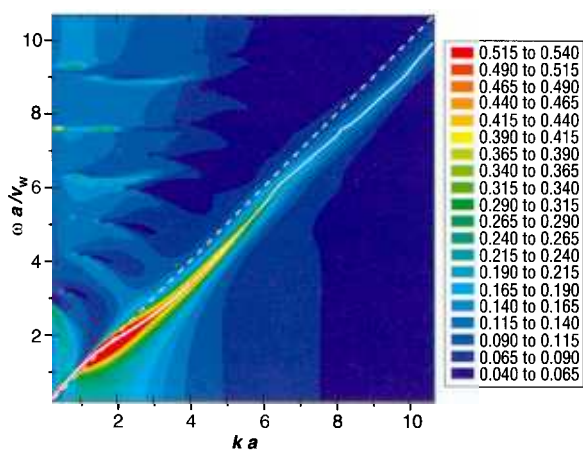
This technique is not restricted to scalar waves; the full elastic wave equation can be used for the solid spheres following the same method. This calculation requires knowledge of the longitudinal and transverse velocities in the solid and the longitudinal velocity in the fluid, as well as the densities of the solid and the fluid and the volume fraction of the solid spheres. Using these known parameters for our sample, we calculated the frequency and wave vector dependence of the spectral function (Fig. 2). For comparison with our experimental data, we used the dispersion curve obtained from the peaks of the spectral function to calculate the phase velocity of the glass-bead suspension from the ratio of frequency to wave vector, averaging over the 10% variation in our bead size. Excellent agreement was obtained (Fig. 1A). We also calculated the group velocity by numerically differentiating the dispersion curve (Fig. 1B) and again obtained excellent quantitative

agreement with our data, capturing the full dispersion in  $v_g$  observed experimentally.

Further insight into the origin of the dispersion in the group velocity was obtained by measuring the scattering mean free path  $l_s$  (Fig. 3). It was determined from the attenuation of the ballistic pulse using the definition  $I(L) = I_0 e^{-L/l_s}$ , where  $I(L)$  is the transmitted intensity for a sample of thickness  $L$  and  $I_0$  is the incident intensity. Independent measurements (11) of the diffusely propagating sound have shown that the absorption length is at least 20 times larger than  $l_s$ , confirming that the observed attenuation of the ballistic pulse is the result of scattering, not absorption. The relatively large values of  $l_s$  at low  $k_w a$  reflect the rapid decrease in scattering as the low-frequency Rayleigh regime is approached. Pronounced dips in  $l_s$  occur at  $k_w a \approx 2$  and 7; these dips correspond almost exactly to the minima observed in the measured group velocity. The origin of these dips can be determined from a calculation of the total scattering cross section of a coated sphere, which is obtained from the imaginary part of the self-energy by using the optical theorem (4). These calculations are in good agreement with the data (Fig. 3), allowing the dips in  $l_s$  to be identified with resonances in the total scattering cross section of a single sphere in the medium. These resonances must also be the origin of the dispersion in  $v_g$ .

The most surprising feature of the data is that a pulse propagates through the highly dispersive medium at such slow velocities; thus, the transmitted pulse is strongly affected by the scatterers but nevertheless maintains temporal and spatial coherence with the incident pulse. The agreement of the calculated group velocity with the data suggests a physical picture for this behavior. When the scattering is strong, each scatterer must sense the scattered waves from the other particles. As a result, the embedding medium becomes renormalized, an effect that is analogous to the shifting of two

**Fig. 2.** Magnitude of the spectral function plotted as a function of normalized frequency  $\omega a/v_w$  and wave vector  $ka$ . The magnitude is indicated by the colors, with red being high and blue being low (the scale bar is in arbitrary units). The dispersion curve, defined by the peaks, is accurately determined because the widths of the peaks are substantially less than their central frequencies. This dispersion curve (solid white line) falls below the dispersion curve for pure water (dashed line) over most of the frequency range. These calculations involve no adjustable parameters.



**Fig. 3.** Scaled scattering mean free path  $l_s/a$  as a function of normalized frequency  $k_w a$ . The curve shows a calculation based on the scattering cross section of a single coated sphere.

quantum mechanical resonances when they are strongly coupled. The amount of this renormalization depends on the strength of scattering; the larger the scattering, the larger the adjustment. Moreover, it is physically plausible that as the concentration of scatterers increases, the material properties of the renormalized effective embedding medium approach those of the scatterers themselves. Consequently, the individual scattering resonances inevitably become leaky and weakened, as the effective contrast between a scatterer and the embedding medium diminishes. In fact, in the limit of the embedding medium having the same properties as the scatterers, the resonances must vanish entirely. It is precisely this effective renormalization of a strongly scattering medium that is sensed by the coherent group velocity. In our calculation, the coated sphere also possesses these scattering resonances, and these are modified by the coupling to the embedding medium. Our procedure effectively identifies the frequencies and wave vectors of the minima in these coupled scattering resonances; these correspond to the dispersion curve. This approach indicates directly how a propagating wave of frequency  $\omega$  is forced by the renormalization of the embedding medium to select the wave vector  $k$  that allows it to propagate through the medium with the least scattering. Hence, the physical origin of the remarkably low velocities of ballistic propagation lies in the renormalization of the effective medium by strong resonant scattering, as correctly described by our theoretical model. Finally we note that, although our experimental and theoretical approach has focused on acoustic waves, we expect the same features to be observed for any form of classical wave propagating in disordered materials; thus, electromagnetic radiation, such as light or microwaves, should show similar behavior, and confirmation of this would provide an important generalization of our results.

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